Sum-difference identites

When you want to find the sine or cosine of the sum or difference of two angles, you can use the sum-difference identities. For two angles θ and α, the sum-difference identities are

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

If we can express an angle as the sum or difference of two angles θ and α whose sines and cosines we know exactly, we can use the sum-difference identities for the sine and cosine functions to find the exact values of the sine and cosine of the sum or difference of θ and α.

Example

Find the exact values of $\cos(\pi/12)$, $\sin(\pi/12)$, $\cos(7\pi/12)$, and $\sin(7\pi/12)$.

If we look at a unit circle, we can see that angles at multiples of $\pi/12$ aren't given. But we can rewrite $\pi/12$ as

$$\frac{\pi}{12} = \frac{(4-3)\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$$

By the difference identity for the cosine function,

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$$\cos\frac{\pi}{12} = \left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{12} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

By the difference identity for the sine function,

$$\sin\frac{\pi}{12} = \left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) - \left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\sin\frac{\pi}{12} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

We also wouldn't normally know the values of sine and cosine at $7\pi/12$, but we can rewrite $7\pi/12$ as

$$\frac{7\pi}{12} = \frac{(4+3)\pi}{12} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$$

By the sum identity for the cosine function,

$$\cos\frac{7\pi}{12} = \left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) - \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

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$$\cos\frac{7\pi}{12} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

By the sum identity for the sine function,

$$\sin\frac{7\pi}{12} = \left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right) + \left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\sin\frac{7\pi}{12} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\sin\frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Let's do a slightly different example.

Example

Suppose θ is an angle in the fourth quadrant whose cosine is $2/\sqrt{5}$, and α is an angle in the second quadrant whose sine is $1/5$. Find the exact values of $\sin\theta$, $\cos\alpha$, $\cos(\theta-\alpha)$, and $\sin(\theta+\alpha)$.

We'll rewrite the basic Pythagorean identity as

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$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Substituting $2/\sqrt{5}$ for $\cos\theta$, we get

$$\sin^2\theta = 1 - \left(\frac{2}{\sqrt{5}}\right)^2$$

$$\sin^2\theta = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\sin\theta = \pm\sqrt{\frac{1}{5}}$$

Since θ is in the fourth quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $1/5$ for $\sin\alpha$, we get

$$\cos^2\alpha = 1 - \left(\frac{1}{5}\right)^2$$

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$$\cos^2\alpha = 1 - \frac{1}{25} = \frac{24}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{24}{25}}$$

Since α is in the second quadrant, we know that $\cos\alpha$ is negative. So we can ignore the positive value and say that

$$\cos\alpha = -\sqrt{\frac{24}{25}}$$

$$\cos\alpha = -\frac{\sqrt{24}}{\sqrt{25}}$$

$$\cos\alpha = -\frac{2\sqrt{6}}{5}$$

By the difference identity for the cosine function,

$$\cos(\theta - \alpha) = \cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha)$$

$$\cos(\theta - \alpha) = \left(\frac{2}{\sqrt{5}}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1}{5}\right)$$

$$\cos(\theta - \alpha) = -\left(\frac{4\sqrt{6}}{5\sqrt{5}}\right) - \left(\frac{1}{5\sqrt{5}}\right)$$

$$\cos(\theta - \alpha) = -\left(\frac{4\sqrt{6}+1}{5\sqrt{5}}\right)$$

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By the sum identity for the sine function,

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)=\left(-\frac{1}{\sqrt{5}}\right)\left(-\frac{2\sqrt{6}}{5}\right)+\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{5}\right)$$

$$\sin(\theta+\alpha)=\frac{2\sqrt{6}}{5\sqrt{5}}+\frac{2}{5\sqrt{5}}$$

$$\sin(\theta+\alpha)=\frac{2\left(\sqrt{6}+1\right)}{5\sqrt{5}}$$

We can use the sum and difference identities for cosine and sine in order to find the sum-difference identities for the tangent function, which are

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Example

Find the exact values of $\tan(\theta+\alpha)$ and $\tan(\theta-\alpha)$ if θ is an angle in the third quadrant whose cosine is $-2/\sqrt{7}$ and α is an angle in the first quadrant whose cosine is $4/7$.

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Before we can find the values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$, we need to find the values of $\sin\theta$ and $\sin\alpha$. We'll rewrite the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

and then substitute $-2/\sqrt{7}$ for $\cos\theta$, we get

$$\sin^2\theta = 1 - \left(-\frac{2}{\sqrt{7}}\right)^2$$

$$\sin^2\theta = 1 - \frac{4}{7} = \frac{3}{7}$$

$$\sin\theta = \pm\sqrt{\frac{3}{7}}$$

Since θ is in the third quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta = -\sqrt{\frac{3}{7}}$$

$$\sin\theta = -\frac{\sqrt{3}}{\sqrt{7}}$$

Now we need to find the value of $\sin\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$

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Substituting $4/7$ for $\cos\alpha$, we get

$$\sin^2\alpha = 1 - \left(\frac{4}{7}\right)^2$$

$$\sin^2\alpha = 1 - \frac{16}{49} = \frac{33}{49}$$

$$\sin\alpha = \pm\sqrt{\frac{33}{49}}$$

$$\sin\alpha = \pm\frac{\sqrt{33}}{7}$$

Since α is in the first quadrant, we know that $\sin\alpha$ is positive. So we can ignore the negative value and say

$$\sin\alpha = \frac{\sqrt{33}}{7}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{3}}{\sqrt{7}}}{-\frac{2}{\sqrt{7}}} = \frac{\sqrt{3}}{2}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{\sqrt{33}}{7}}{\frac{4}{7}} = \frac{\sqrt{33}}{4}$$

By the sum identity for the tangent function,

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$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{33}}{4}}{1-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+1\sqrt{33}}{4}}{1-\frac{\sqrt{99}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{33}}{4}}{1-\frac{\sqrt{9(11)}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{3}\sqrt{11}}{4}}{1-\frac{3\sqrt{11}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}\left(2+\sqrt{11}\right)}{4}}{\frac{8(1)-1\left(3\sqrt{11}\right)}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}\left(2+\sqrt{11}\right)}{4}}{\frac{8-3\sqrt{11}}{8}}$$

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$$\tan(\theta+\alpha)=\frac{2\sqrt{3}\left(2+\sqrt{11}\right)}{8-3\sqrt{11}}$$

By the difference identity for the tangent function,

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sqrt{3}}{2}-\frac{\sqrt{33}}{4}}{1+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta-\alpha)=\frac{\frac{2\sqrt{3}-1\sqrt{33}}{4}}{1+\frac{\sqrt{99}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\frac{2\sqrt{3}-\sqrt{33}}{4}}{1+\frac{\sqrt{9(11)}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\frac{2\sqrt{3}-\sqrt{3}\sqrt{11}}{4}}{1+\frac{3\sqrt{11}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sqrt{3}\left(2-\sqrt{11}\right)}{4}}{\frac{8(1)+1\left(3\sqrt{11}\right)}{8}}$$

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$$\tan(\theta-\alpha)=\frac{\frac{\sqrt{3}\left(2-\sqrt{11}\right)}{4}}{\frac{8+3\sqrt{11}}{8}}$$

$$\tan(\theta-\alpha)=\frac{2\sqrt{3}\left(2-\sqrt{11}\right)}{8+3\sqrt{11}}$$

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